



Meshless Methods

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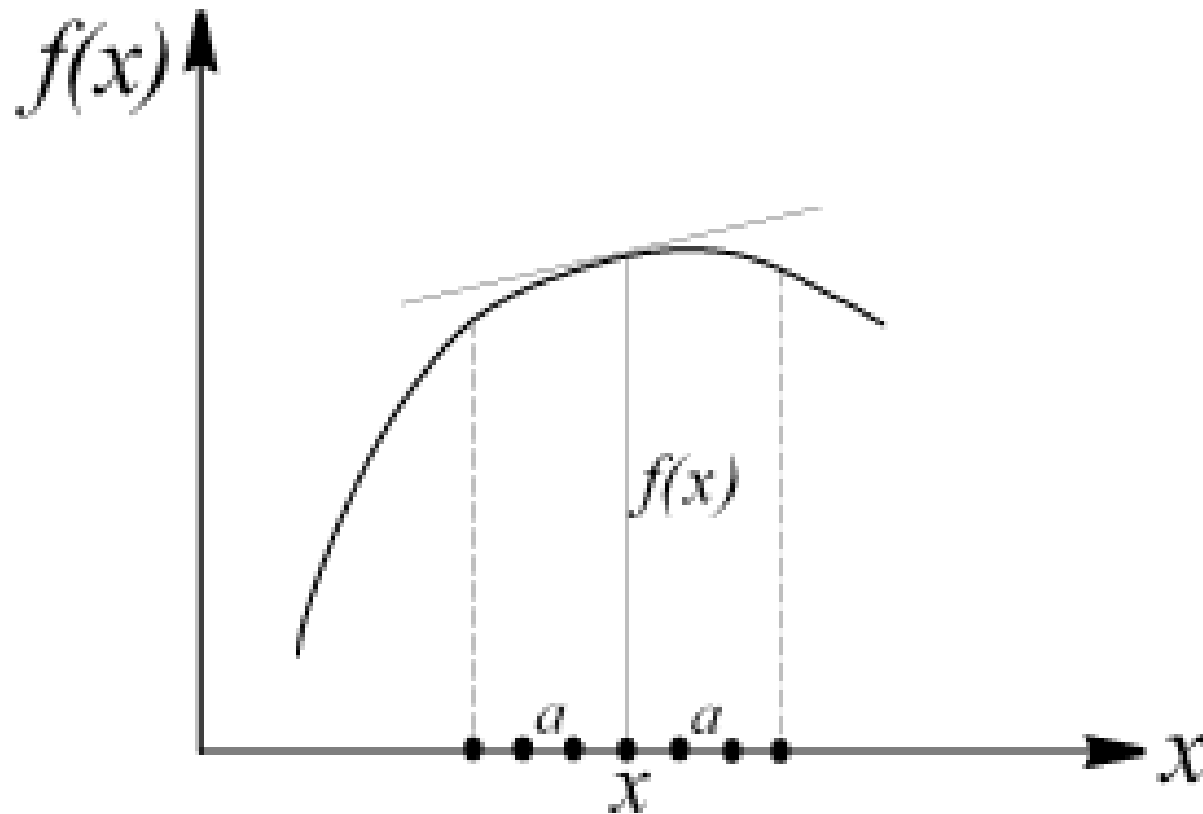
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Approximation Methods:

- There are three general approaches towards approximating a continuous function at x using the information in its vicinity:
 1. Finite differential representation.
 2. Finite series representation.
 3. Finite integral representation.

Approximation Methods:

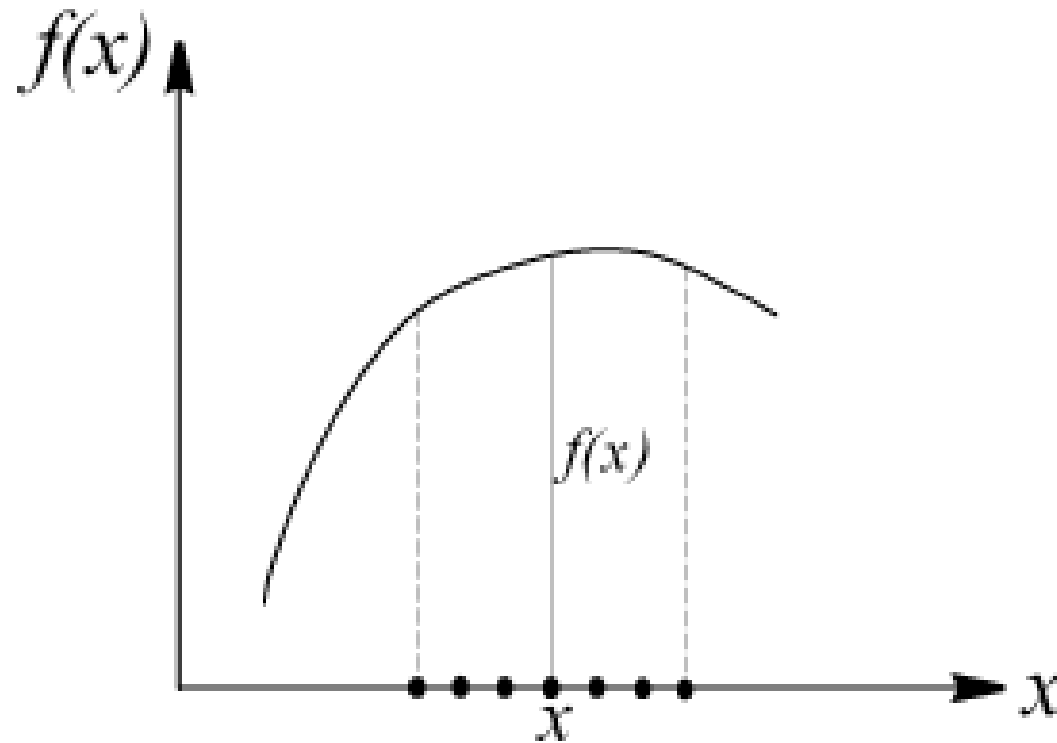
1. Finite differential representation.



$$f(x) \cong f(x_0) + f'(x_0)(x-a) + \frac{1}{2} f''(x_0)(x-a)^2 + \dots$$

Approximation Methods:

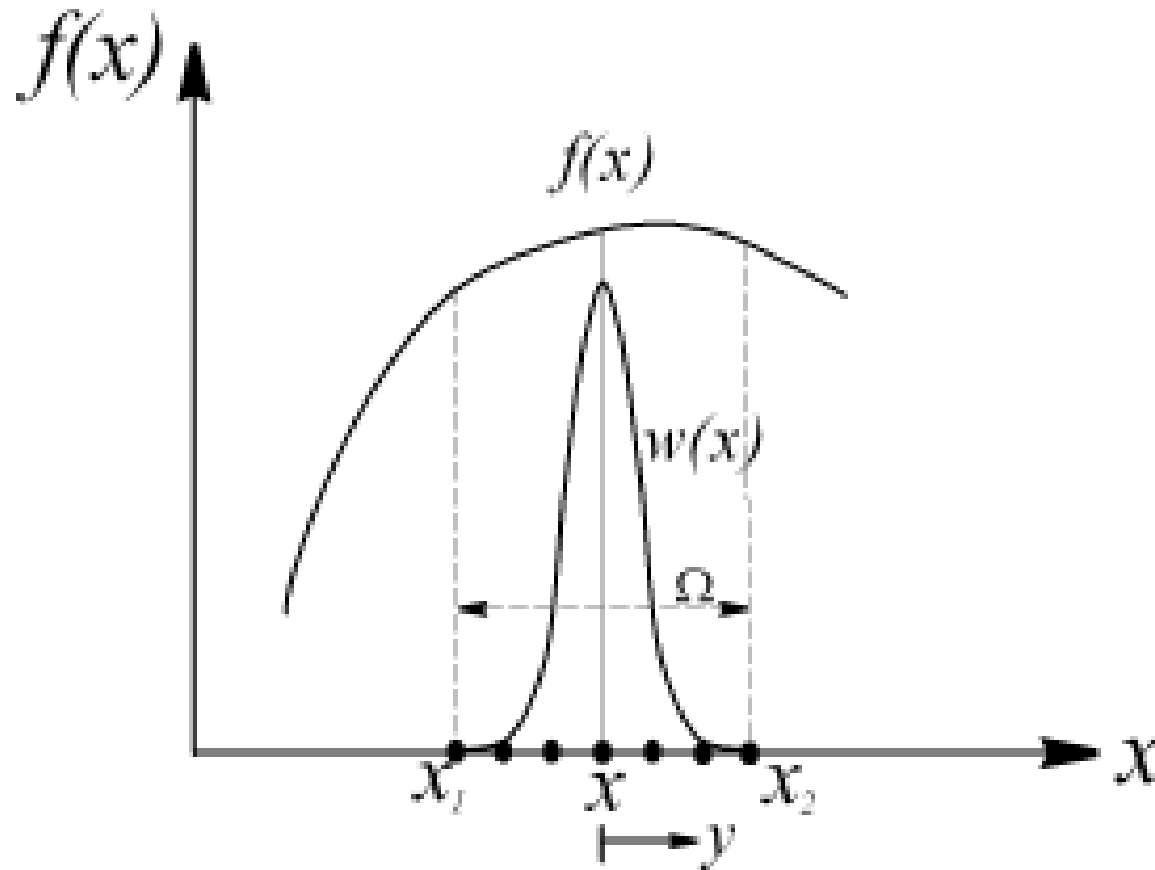
2. Finite series representation.



$$f(x) \cong a_0 + a_1 p_1(x) + a_2 p_2(x) + \dots$$

Approximation Methods:

3. Finite integral representation.



$$f(x) \cong \int_{x_1}^{x_2} f(y)W(x-y)dy$$

Approximation Methods:

1. Finite differential representation.
 1. Finite difference method for regular grids
 2. Generalized finite difference method for irregular grids
 3. Finite point method
 4. ...

Approximation Methods:

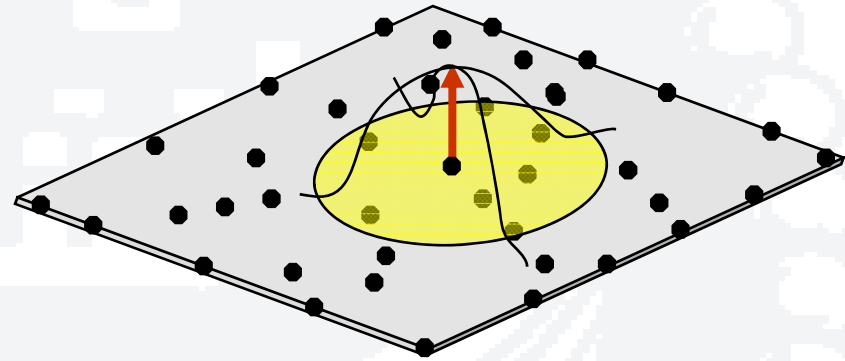
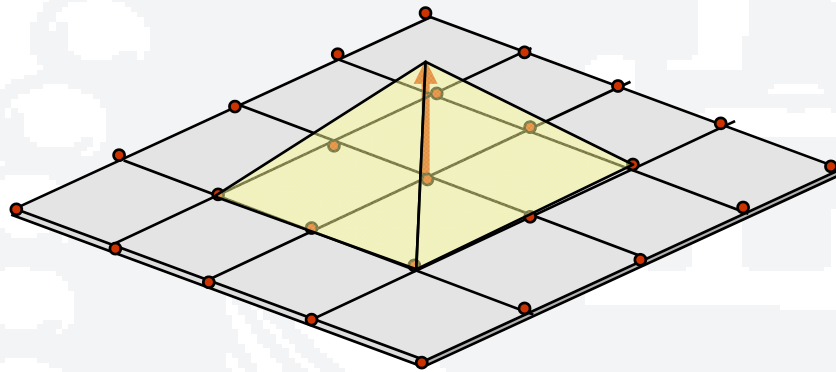
2. Finite series representation.

1. Finite element method (FE)
2. Least square method (LS)
3. Weighted least squares method (WLS)
4. Moving least squares method (MLS)
5. Point interpolation method (PIM)
6. Partition of unity method (PU)
7. ...

Approximation Methods:

3. Finite integral representation.
 1. Smoothed particle hydrodynamics (SPH)
 2. Reproducing kernel particle method (RKPM)
 3. Corrected smoothed particle method (CSPM)
 4. Generalized smoothed particle hydrodynamics (GSPH)
 5. Moving least square reproducing kernel particle method (MLSRKPM)
 6.

Approximation Methods:



Truncated Taylor Series (TTS):

- One of the simplest approximation methods.
- TTS has been successfully adopted within an irregular finite difference approximation.

Truncated Taylor Series (TTS):

- MacLauren approximation of the continuous function $u(x)$

$$u(x) = u(0) + xu_{,x}(0) + yu_{,y}(0) + \frac{1}{2}x^2u_{,xx}(0) + \frac{1}{2}y^2u_{,yy}(0) + \dots$$

- or

$$u^h(x) = u(0) + xu_{,x}(0) + yu_{,y}(0) + \dots + O(h^{p+1})$$

- where

p = complete polynomial order

h = average nodal distance

O = order of error

Truncated Taylor Series (TTS):

- Theoretically, $h \downarrow$ or $p \uparrow$ reduce the order of error O .
- Replacing the derivatives by unknown coefficients results in:

$$u^h(x) = a_0 + a_1x + a_2y + a_3x^2 + a_4xy + a_5y^2 + \dots$$

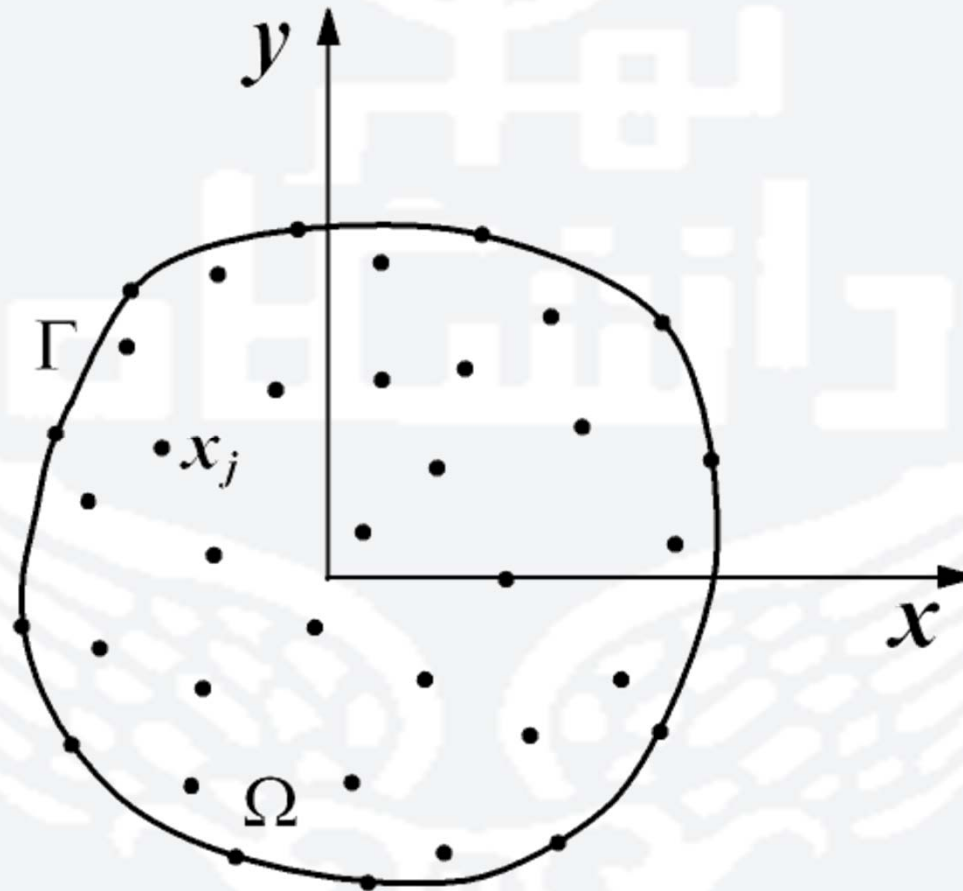
$$u^h(x) = \sum_{k=1}^m P_k(x) a_k = P^T(x) a$$

- where

$$P(x) = (1, x, y, x^2, xy, y^2, \dots)^T \quad : \text{basis function}$$

$$a = (a_0, a_1, a_2, a_3, a_4, \dots)^T \quad : \text{vector of unknowns}$$

Truncated Taylor Series (TTS):



Irregular distribution of nodal points

Truncated Taylor Series (TTS):

- Coefficients a_k are determined from:

$$u^h(x_j) = u(x_j) = \bar{u}_j, \quad j = 1, \dots, n$$

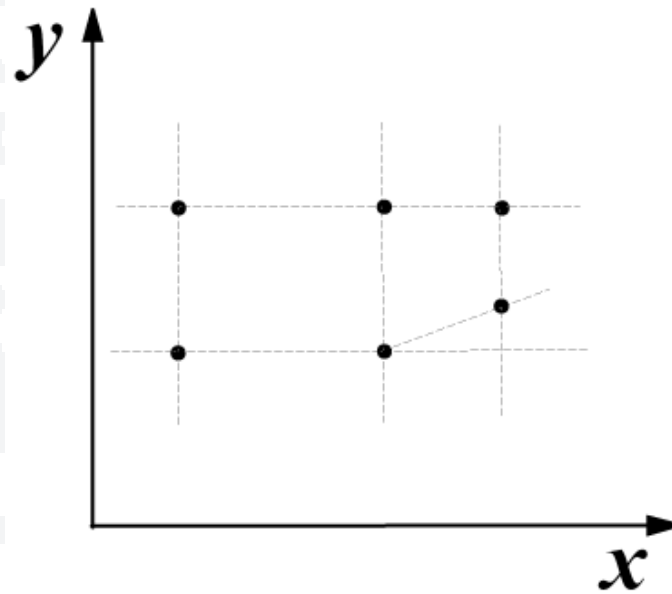
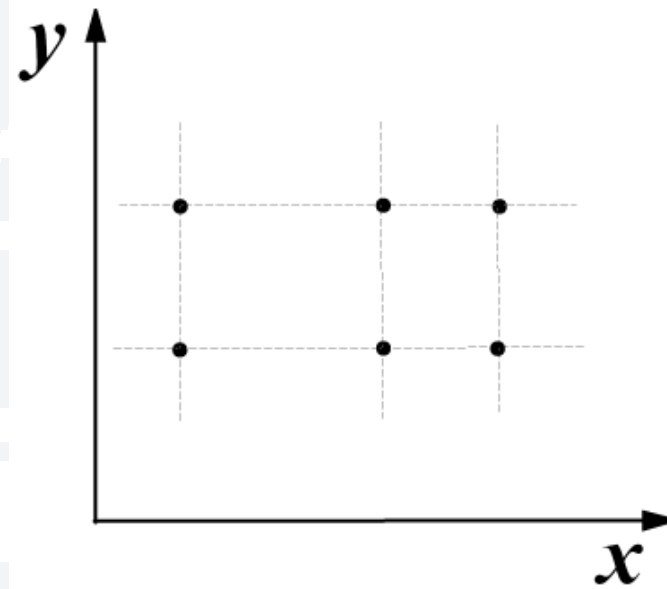
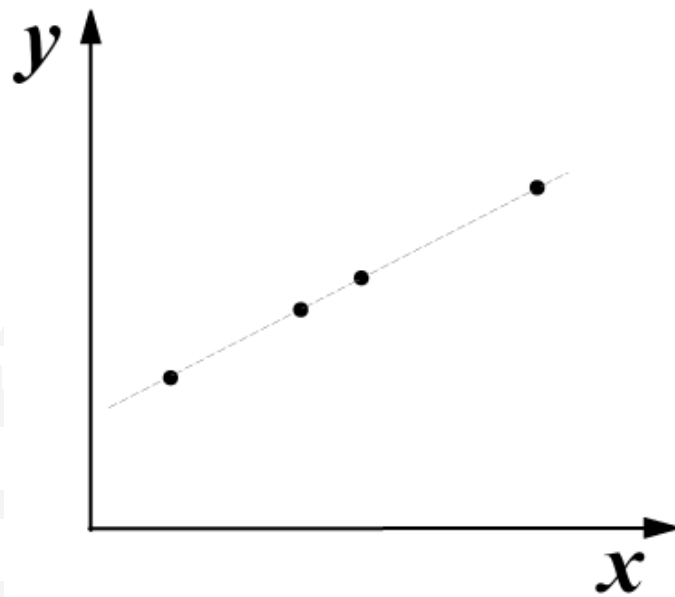
- or

$$P^T(x_j)a = \bar{u}_j, \quad j = 1, \dots, n$$

- Note that:

- A unique solution is guaranteed if $n=m$ (equal number of nodes and basis function terms)
- Singularity is possible for regular nodal distribution
- Ill conditioning is also possible for many near regular nodal distribution.

Truncated Taylor Series (TTS):



Truncated Taylor Series (TTS):

- The classical form of approximation in terms of the shape function:

$$u^h(x) = \sum_{j=1}^n \phi_j(x) \bar{u}_j$$

- with
 - n : total number of nodes
 - $\phi_j(x)$: shape function associated with node j
 - $\bar{u}_j = u(x_j)$: value of the function at node j
 - $x = (x, y, z)^T$: position of any point within the domain

Lagrange Interpolation (LI):

- In the Lagrange interpolation, the shape functions are explicitly defined in terms of the nodal positions.
- LI is an interpolation technique.
- Finite element shape functions are usually constructed from the lagrange interpolation.
- In 1D, LI is equivalent to TTS.

Lagrange Interpolation (LI):

$$u^h(x) = \sum_{j=1}^n \phi_j(x) \bar{u}_j$$

- For 1D problems, quadratic LI interpolation based on three nodal values takes the following form,

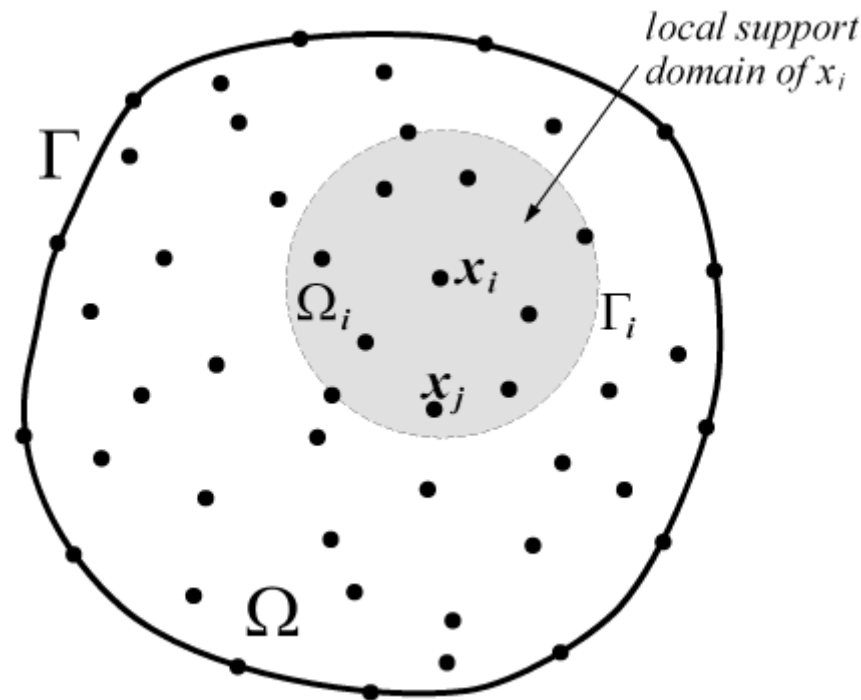
$$\phi_1(x) = \frac{(x - x_3)(x - x_2)}{(x_1 - x_3)(x_1 - x_2)}$$

$$\phi_2(x) = \frac{(x - x_3)(x - x_1)}{(x_2 - x_3)(x_2 - x_1)}$$

$$\phi_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)}$$

Least Square Approximation (LS):

- Least square methods have been proposed to avoid singularity or ill conditioning in truncated Taylor series.
- LS uses larger number of nodes around a specific node than the number of basis functions ($n > m$).



$$j = 1, \dots, n_i$$

$$n_i > m$$

Least Square Approximation (LS):

- As a result, complete satisfaction of

$$P^T(x_j)a = \bar{u}_j \quad , j = 1, \dots, n_i$$

is no longer possible.

- Instead, the average error of approximation is minimized:

$$\Pi = \sum_{j=1}^{n_i} [u^h(x_j) - \bar{u}_j]^2 = \sum_{j=1}^{n_i} [P^T(x_j)a - \bar{u}_j]^2$$

- Minimization of Π with respect to the unknown a_k

$$\frac{\partial \Pi}{\partial a_k} = 0 \quad , k = 1, \dots, m$$

Least Square Approximation (LS):

- A system of $m \times m$ simultaneous equations are then obtained.

$$Aa = B\bar{U}$$

- where

$$A = \sum_{j=1}^{n_i} P(x_j)P^T(x_j)$$

$$B = [P(x_1), \dots, P(x_{n_i})]$$

- If solved for the constant parameter a ,

$$a = A^{-1}B\bar{U}$$

Least Square Approximation (LS):

- The final LS approximation is obtained:

$$u^h(x) = P^T(x)a = P^T(x)A^{-1}B\bar{U}$$

- And the LS shape function

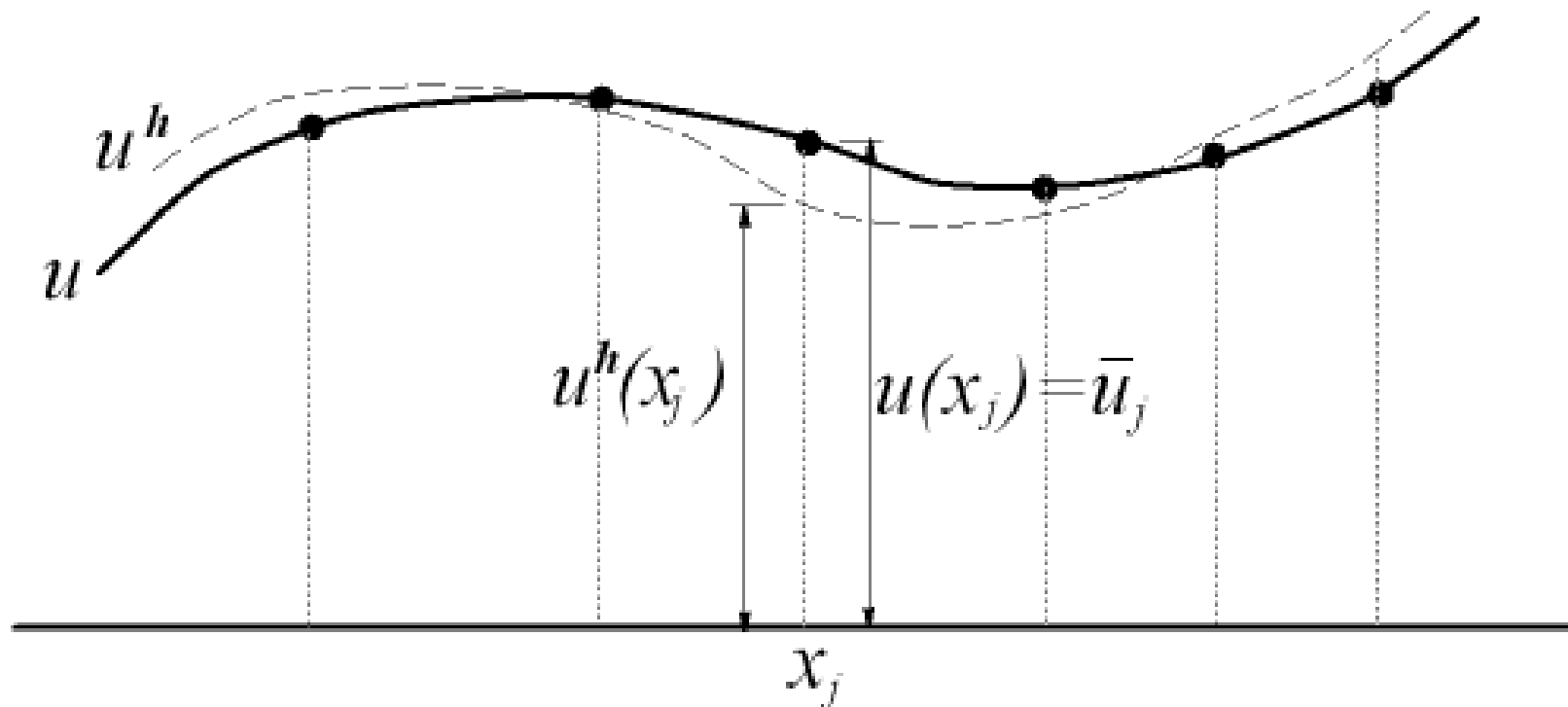
$$u^h(x) = \phi(x)\bar{U} = \sum_{j=1}^{n_i} \phi_j(x)\bar{u}_j$$

$$\phi(x) = P^T(x)A^{-1}B$$

Least Square Approximation (LS):

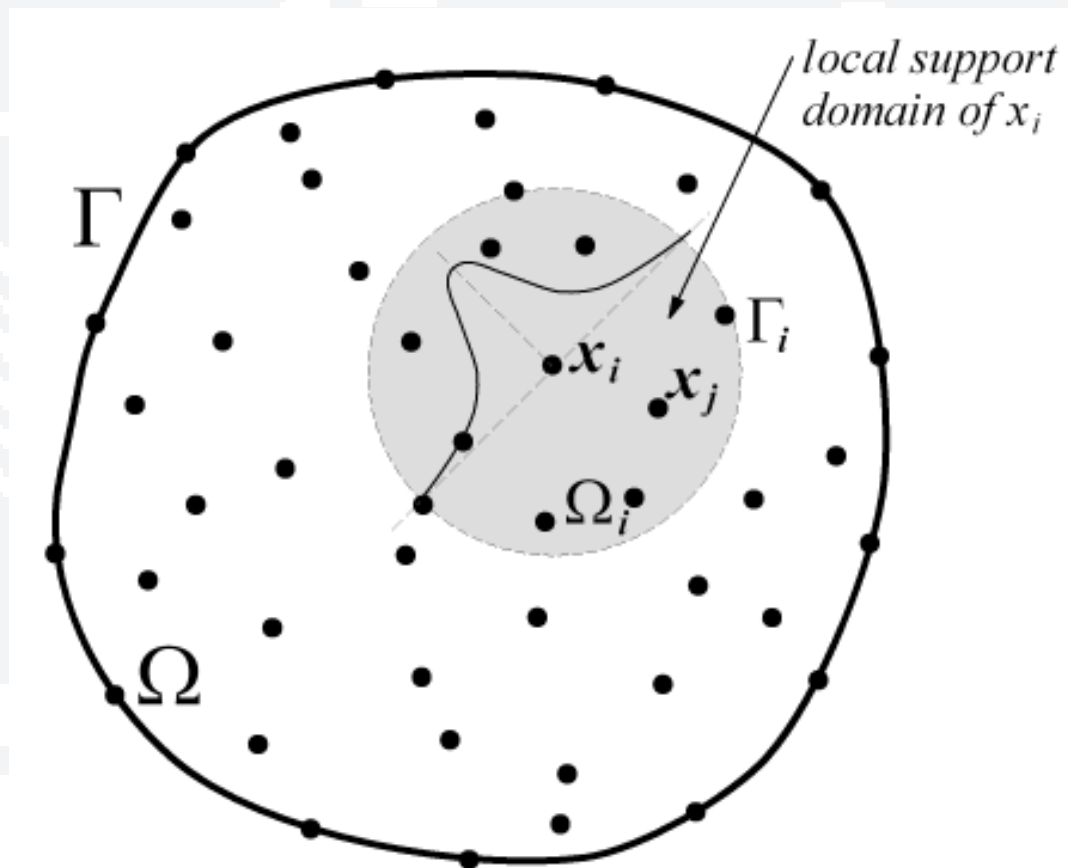
- LS is clearly an approximation rather than an interpolation:

$$u^h(x_j) \neq \bar{u}_j$$



Weighted Least Square (WLS):

- The weighted least square technique (WLS) is used to improve the approximation around the central/polar point x_i by introduction of a weight function $W(x_j - x_i)$.



Weighted Least Square (WLS):

- The weighted average error of approximation is defined as:

$$u^h(x) = P^T(x)a$$

$$\Pi = \sum_{j=1}^{n_i} W_i(x_j - x_i) [u^h(x_j) - \bar{u}_j]^2$$

- and then

$$u^h(x_j) = P^T(x_j)a$$

$$\Pi = \sum_{j=1}^{n_i} W_i(x_j - x_i) [P^T(x_j)a - \bar{u}_j]^2$$

- Minimization of Π with respect to the unknown a_k

$$\frac{\partial \Pi}{\partial a_k} = 0, k = 1, \dots, m$$

Weighted Least Square (WLS):

- The following system of $m \times m$ simultaneous equations are then obtained.

$$Aa = B\bar{U}$$

- where

$$A = \sum_{j=1}^{n_i} W_i(x_j - x_i)P(x_j)P^T(x_j)$$

$$B = [W_i(x_1 - x_i)P(x_1), \dots, W_i(x_{n_i} - x_i)P(x_{n_i})]$$

- If solved for the constant parameter a ,

$$a = A^{-1}B\bar{U}$$

Weighted Least Square (WLS):

- The final WLS approximation is obtained:

$$u^h(x) = P^T(x)a = P^T(x)A^{-1}B\bar{U}$$

- and the WLS shape function

$$u^h(x) = \phi(x)\bar{U} = \sum_{j=1}^{n_i} \phi_j(x)\bar{u}_j$$

$$\phi(x) = P^T(x)A^{-1}B$$

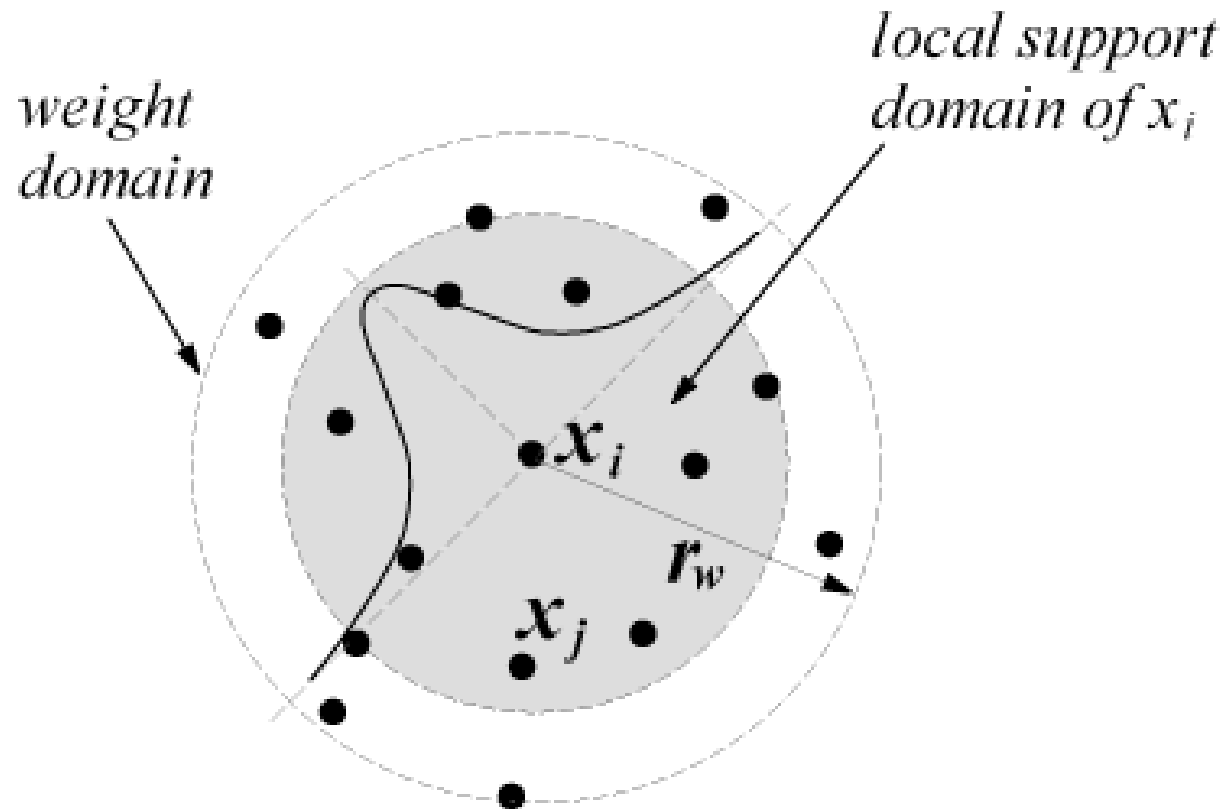
- WLS is not an interpolation:

$$u^h(x_j) \neq \bar{u}_j$$

Weighted Least Square (WLS):

- W has the maximum value at the central node x_i .
- W is reduced by the distance from the central node.
- The size of the weight domain must be equal or larger than the size of the support domain.
- W should preferably vanish on the boundaries of the support domain.
- Bell shaped functions are frequently used.

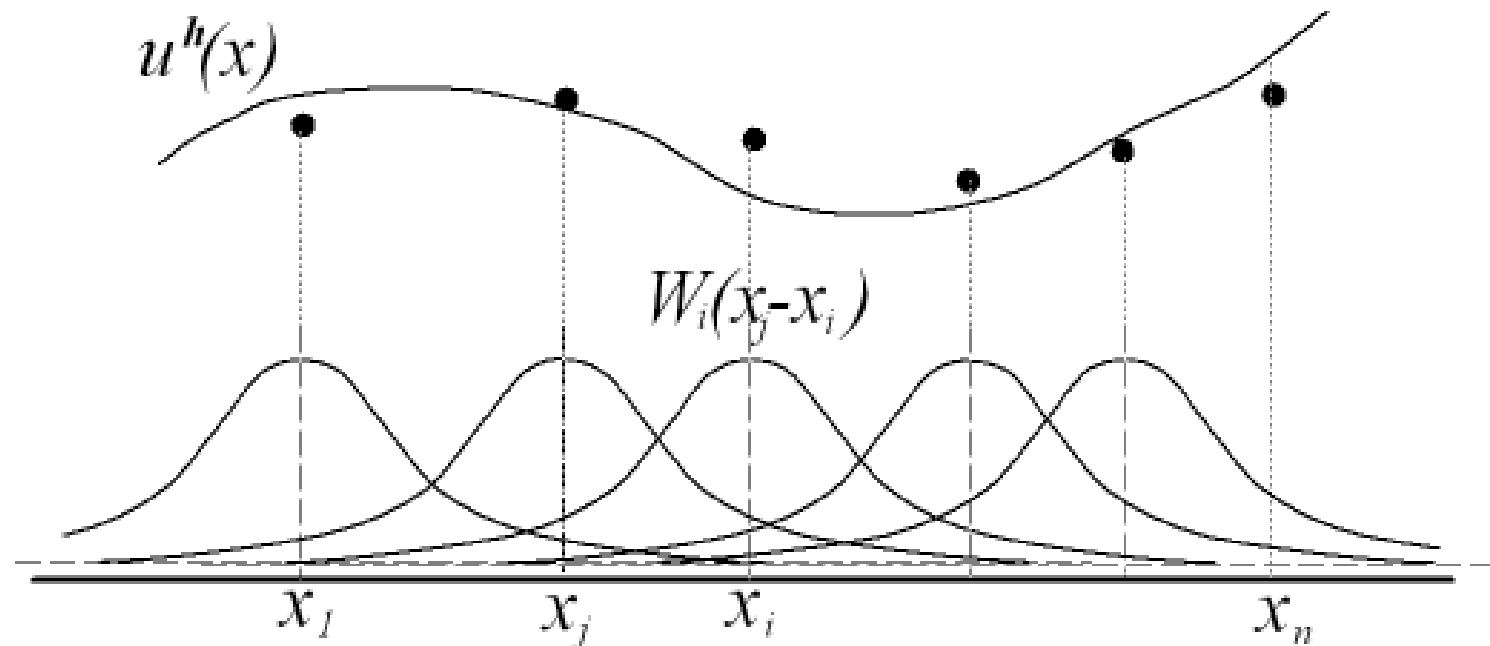
Weighted Least Square (WLS):



$$W_i(r) = W_i(x - x_i)$$

$$r^2 = (x - x_i) \cdot (x - x_i)$$

Weighted Least Square (WLS):



Weighted Least Square (WLS):

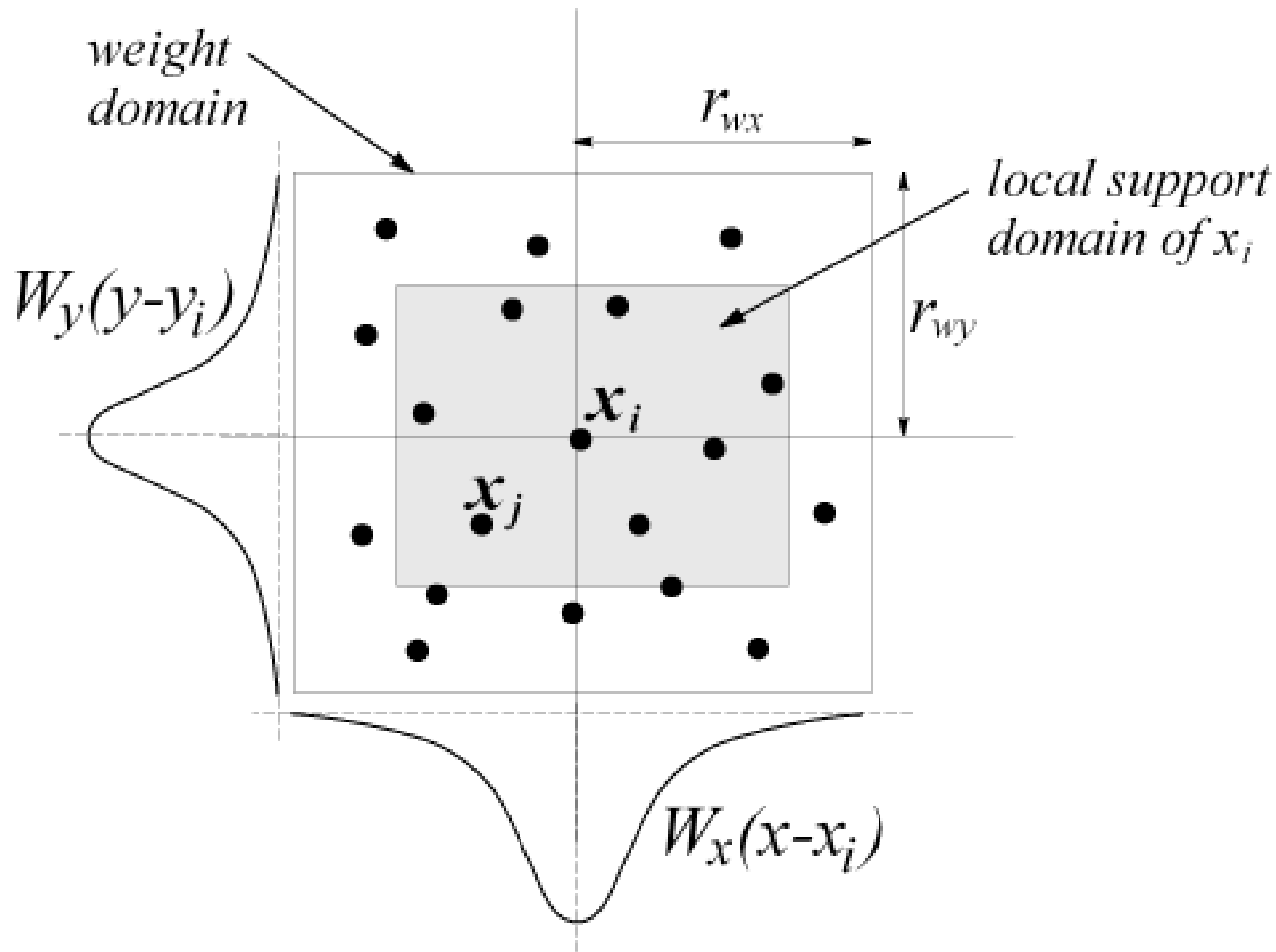
$$W_i(r) = \begin{cases} \frac{e^{-(r/c)^2} - e^{-(r_m/c)^2}}{1 - e^{-(r_m/c)^2}} & r \leq r_w \\ 0 & r > r_w \end{cases}$$

$$W_i(r) = \begin{cases} 1 - 6s^2 + 8s^3 - 3s^4 & r \leq r_w \\ 0 & r > r_w \end{cases}, s = \frac{r}{r_w}$$

$$W_i(r) = \begin{cases} 1 - 3s + 2s^2 & r \leq r_w \\ 0 & r > r_w \end{cases}, s = \frac{r}{r_w}$$

$$W_i(r) = \begin{cases} (1 - s^2)^n, n \geq 2 & r \leq r_w \\ 0 & r > r_w \end{cases}, s = \frac{r}{r_w}$$

Weighted Least Square (WLS):



$$W(x - x_i) = W_x(x - x_i)W_y(y - y_i)$$

